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NUSC Technical Report 8403
6 October 1988

Fourth-Order Spectra of Mixture and Modulated Processes

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This research was conducted under NUSC project No. A60070, "Acoustic Signal Processing." The NUSC Program Manager was John Martins, Jr., (Code 2111), and Principal Investigator was Dr. Roger F. Dwyer (Code 3314). This work was sponsored by Capt. R. Fitch of ONT. The program manager at ONT was Dr. T. Warfield.

The technical reviewer for this report was William Strusinski (Code 2111).

Reviewed and Approved: 6 October 1988

Donald W. Counsellor

ASW Systems Department

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT APPROVED FOR PUBLIC RELEASE; DISTRIBUTION IS UNLIMITED
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 8403	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION NAVAL UNDERWATER SYSTEMS CTR 6b. OFFICE SYMBOL (If applicable) CODE 3314	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) NEW LONDON LABORATORY NEW LONDON, CT 06320	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
FOURTH-ORDER SPECTRA OF MIXTURE AND MODULATED PROCESSES

12. PERSONAL AUTHOR(S)
ROGER F. DWYER

13a. TYPE OF REPORT	13b. TIME COVERED FROM ____ TO ____	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
		6 OCT 1988	

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	FOURTH-ORDER SPECTRUM, DOPPLER SPREAD, MODULATION, SONAR WAVEFORMS, MIXTURE PROCESS, AUTOCORRELATION, SPECTRUM

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

For non-Gaussian processes higher-order spectra contain information that usually is not contained in second-order spectra. This hypothesis shall be proven for a special case of the trispectrum using mixture and modulated processes. Specifically, the fourth-order cumulant spectrum is shown to differentiate between a Gaussian process and a non-Gaussian mixture process. Mixture processes often represent physical phenomena arising in sonar and radar applications. For example, the reflected components of an active transmission can be composed of a mixture of two sinusoids due to transient backscatter of creeping waves. The resonances of this phenomenon are a means to target classification. However, resonances do not represent a unique way of classifying the phenomenon, since, a sum of sinusoids can have an identical spectrum. But, on the other hand, a sum of sinusoids do not represent the creeping wave phenomenon. It is demonstrated in the report that the fourth-order cumulant spectrum can differentiate between a sum of sinusoids and a mixture of sinusoids.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL R. F. Dwyer	22b. TELEPHONE (Include Area Code) (203) 440-4511 22c. OFFICE SYMBOL 3314

DD FORM 1473, 84 MAR

83 APR edition may be used until exhausted.
All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

No. 19. ABSTRACT (Cont'd.)

Modulated processes also arise in sonar, radar, and communications applications. For example, when a long pulse is transmitted, reflectivity variations and target dynamics will often amplitude modulate the return waveform. Using a coded pulse train, it is shown in the report that the fourth-order cumulant spectrum can extract target information from an amplitude modulated return whereas the information in the second-order spectrum is limited by the modulating process.

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EXECUTIVE SUMMARY

The fourth-order spectrum represents a new way of extracting information from data. Used in conjunction with the second-order spectrum (or power spectrum) the fourth-order spectrum reveals otherwise hidden relationships that are important for classification of transient signals in passive sonar and for classification of resonances in active sonar. In addition, the fourth-order spectrum can extract range and Doppler information from an active sonar return under conditions which render the conventional spectrum useless. Moreover, the potential for signal-to-noise ratio improvements in Gaussian noise environments is high. Theoretically, the fourth-order spectrum eliminates all additive Gaussian noise from its final result. The practicality of this theoretical fact is currently being considered.

The drawbacks in using the fourth-order spectrum in practice are: 1) it requires more processing capacity, and 2) it will require training in order to properly interpret the results, unless the detection and classification functions are automated. However, these drawbacks have been minimised in this report by the utilisation of a special case of the fourth-order spectrum. Nevertheless, all the advantages of the fourth-order spectrum discussed above have been maintained.

The following points are discussed in more detail in the report:

1) A new class of multidimensional non-Gaussian density functions are introduced. This class represents physically meaningful signals, since, independent data are not required. As a special case, the class reduces to a multidimensional Gaussian density, so a wide variety of density functions can be represented. In the report fourth-order spectra were obtained for the class of multidimensional non-Gaussian density functions. In this way, it was shown that the fourth-order spectrum could differentiate between a *Gaussian* and a *non-Gaussian* process. This is important because many noise sources in the ocean are Gaussian. For example, ambient noise, reverberation, flow noise, ect., are Gaussian. Whereas, signals are usually non-Gaussian. For example, sinusoids, transients, active sonar transmissions, ect., are all non-Gaussian.

2) An active sonar classification technique relies on backscatter resonances to differentiate target types. But resonances may not give an unique classification, since, many waveforms could have the same resonances. In order to show the potential of the fourth-order spectrum for classification, an idealised example of creeping wave backscatter is discussed. In this example, a sum of two sinusoids and a mixture of two sinusoids have the same spectrum (resonances). However, only the mixture represents the physical phenomenon of creeping wave backscatter. It is shown that the fourth-order spectrum differentiates between a sum and a mixture of sinusoids, and therefore, can aid in active sonar classification.

3) When long pulse trains are transmitted the return echo is usually amplitude modulated from target dynamics, Doppler spreading, or medium effects. Under these conditions, the spectrum can be severly distorted making detection and classification impossible. Using an amplitude modulated coded pulse train, it is shown that the fourth-order spectrum can extract range and Doppler information while for the same conditions the spectrum is useless.

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Fourth-Order Spectra of Mixture and Modulated Processes

1. Introduction

Certain underwater acoustic signals can be modeled as mixture processes. For example, a scattered wave from an extended target will often consist of several subwaves with different characteristics. Determining whether or not a received waveform is from an extended target is an important problem[1,2]. Another example of a mixture process is an intermittency produced by a complex mechanical source[3]. Also, environmental conditions can cause several signals with different doppler frequencies to be mixed together in the received waveform. In all of these examples the fundamental properties of the waveform can be characterised by a mixture process. Other signals are better modeled as modulated processes. Wave propagating in random medium and Doppler spread targets are two examples.

The objectives of the report are: 1) to develop a parameterized mixture process model, similar but more general than reference 4, that characterises the phenomena of interest and extract information from the model by estimating its spectra and higher- order spectra [5,6,7]; and 2) to estimate spectra and higher-order spectra from amplitude modulated processes.

It will be shown in the report that the mixture process is in general a non-Gaussian process. Therefore, higher-order spectra would contain additional information about the non-Gaussian waveform. In our initial application, a measurable higher-order spectrum would be enough to classify the phenomenon as belonging to a mixture process. Specifically, it will be shown that a special case of the trispectrum [8,9] can determine whether or not a mixture process is present, whereas, the second-order spectrum cannot. For this special case, an interesting result for sinusoidal mixture processes is presented and demonstrated by simulation. For the modulated case, examples are presented which show the usefulness of the fourth-order cumulant spectrum in extracting target information from an amplitude modulated return.

The second-order (power) spectrum the and fourth-order spectrum (trispectrum) are special cases of the nth-order spectrum[8,10]. The nth-order spectrum is defined as the Fourier transform of the nth-order cumulant. Therefore, the nth-order spectrum is also called the nth-order cumulant spectrum. For a zero mean and stationary process the second-order and fourth-order cumulant functions are as follows:

<u>second-order</u>

$$C_2(\tau) = E[x(t)x(t+\tau)],$$

<u>fourth-order</u>

$$\begin{aligned} C_4(\tau_1,\tau_2,\tau_3) &= E[x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3)] \\ &\quad - E[x(t)x(t+\tau_1)]E[x(t+\tau_2)x(t+\tau_3)] \\ &\quad - E[x(t)x(t+\tau_2)]E[x(t+\tau_1)x(t+\tau_3)] \\ &\quad - E[x(t)x(t+\tau_3)]E[x(t+\tau_1)x(t+\tau_2)]. \end{aligned}$$

Since the Gaussian expansion is subtracted from the fourth-order moment, the fourth-order cumulant is sero for Gaussian processes. Therefore, as noted by Rosenblatt [5], additive Gaussian noise will not effect the result of the trispectrum.

This report is concerned with a special case of the fourth-order cumulant. Namely, when, $\tau_1 = 0$, and $\tau_2 = \tau_3$. Therefore, the fourth-order cumulant reduces to the following:

$$\begin{aligned} C_{42}(\tau) &= E[x(t)^2 x(t+\tau)^2] - E[x(t)^2]E[x(t+\tau)^2] - 2E[x(t)x(t+\tau)]^2 \\ &= R_2(\tau) - \mathrm{Var}[x(t)]^2 - 2[R(\tau)]^2. \end{aligned} \tag{1.1}$$

If in addition $\tau = 0$, then C_{42} is related to the kurtosis of x. Properties of kurtosis were discussed in previous publications[11,12].

The fourth-order spectrum is therefore,

$$C_{42}(\omega) = \int_{-\infty}^{+\infty} C_{42}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} R_2(\tau) e^{-j\omega\tau} d\tau - 2\pi[\mathrm{Var}[x(t)]]^2 \delta(\omega) - 2\int_{-\infty}^{+\infty} R(\tau)^2 e^{-j\omega\tau} d\tau. \tag{1.2}$$

Another special case of the fourth-order cumulant arises when $\tau_1 = \tau_2 = 0$ or when $\tau_1 = \tau_2 = \tau_3 = \tau$.

For this case,

$$\begin{aligned} C_{43}(\tau) &= E[x(t)^3 x(t+\tau)] - 3E[x(t)^2]E[x(t)x(t+\tau)] \\ &= R_3(\tau) - 3\mathrm{Var}[x(t)]R(\tau). \end{aligned}$$

When the process is Gaussian, $C_{43}(\tau) = 0$. Therefore, its spectrum is also sero for all frequencies.

2. Mixture Processes

The need for multidimensional non-Gaussian processes arise from the definition of the nth-order cumulant spectra. Here non-Gaussian processes will be modeled as non-Gaussian mixture processes. As alluded to in the introduction, these mixture processes are more than mathematical conveniences, because they seem to represent physical processes.

A bivariate mixture probability density is defined as follows:

$$f(x,y) = (1-\lambda)^2 f_1(x,y) + \lambda^2 f_2(x,y) + \lambda(1-\lambda)[f_1(x)f_2(y|x) + f_2(x)f_1(y|x)], \tag{2.1}$$

where, $0 \leq \lambda \leq 1$ and $\int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

The bivariate densities $f_1(x,y)$ and $f_2(x,y)$ can be in general any bivariate densities. However, Gaussian densities are utilized in this report in order to obtain tractable results. But, even when $f_1(x,y)$ and $f_2(x,y)$ are Gaussian the mixture density $f(x,y)$ is in general non-Gaussian. This was demonstrated for univariate densities by Bendat and Piersol [13]. If $f_1(x,y) \neq f_2(x,y)$ and $\lambda \neq 0,1$, then $f(x,y)$ is non-gaussian. The Gaussian densities $f_1(x,y)$ and $f_2(x,y)$ are defined below.

Suppose the marginals are the same in (2.1), i.e., $f_1(x) = f_2(x)$. Then,

$$f(x,y) = (1-\lambda)f_1(x,y) + \lambda f_2(x,y). \tag{2.2}$$

As an application of (2.2), let the mixture process be passed through a hard clipper. It is desired to find the autocorrelation function at the output. The hard clipper is sometimes used in sonar arrays and other applications include neural networks[14]. For the Gaussian case the result is well known[15,16]. However, for the non-Gaussian case no general result is known. But, a general result for the non-Gaussian mixture density (2.2) reduces to the following,

$$R_c(\tau) = (1-\lambda)\frac{2}{\pi}\sin^{-1}[\rho_1(\tau)] + \lambda\frac{2}{\pi}\sin^{-1}[\rho_2(\tau)].$$

If, $\lambda = 0, 1$, then $R_c(\tau)$ reduces to the Gaussian case. Also, the autocorrelation function at the output of more general nonlinearities [17] can be similarly derived.

The form of (2.1) is more general than the bivariate mixture densities discussed in reference 4. This generality represents an improvement, since, (2.1) does not suffer from the disadvantages of the mixture densities of reference 4. For example, if x and y are independent than (2.1) and (2.2) reduce to a product of univariate mixture densities. Also, (2.1) and all special cases, e.g., (2.2), always satisfy the strong mixing condition.

The form of (2.1) can also be extended to a multivariate non-Gaussian mixture density, but this generality is not needed here.

Let,

$$f_1(x,y) = \frac{1}{2\pi}\frac{1}{\sigma_1(1-\rho_1^2)^{1/2}}e^{-\frac{1}{2}\frac{1}{\sigma_1^2(1-\rho_1^2)}(x^2+y^2-2\rho_1 xy)},$$

and

$$f_2(x,y) = \frac{1}{2\pi}\frac{1}{\sigma_2(1-\rho_2^2)^{1/2}}e^{-\frac{1}{2}\frac{1}{\sigma_2^2(1-\rho_2^2)}(x^2+y^2-2\rho_2 xy)},$$

be the mixture densities of (2.1). Then the autocorrelation function,

$$R(\tau) = \int_{-\infty}^{\infty} xyf(x,y)dxdy,$$

reduces to the following,

$$R(\tau) = [(1-\lambda)\rho_1(\tau) + \lambda\rho_2(\tau)][(1-\lambda)\sigma_1^2 + \lambda\sigma_2^2)]. \tag{2.3}$$

The spectrum of (2.3) is

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$$

$$= [(1-\lambda)\sigma_1^2 + \lambda\sigma_2^2]\left[(1-\lambda)\int_{-\infty}^{\infty}\rho_1(\tau)e^{-j\omega\tau}d\tau + \lambda\int_{-\infty}^{\infty}\rho_2(\tau)e^{-j\omega\tau}d\tau\right]. \tag{2.4}$$

On the other hand, the fourth-order moment of (2.1),

$$R_2(\tau) = \int_{-\infty}^{\infty} x^2y^2 f(x,y)dxdy,$$

reduces to the following result,

$$\begin{aligned} R_2(\tau) &= [(1-\lambda)\sigma_1^2 + \lambda\sigma_2^2]^2 \\ &+2[(1-\lambda)\rho_1^2(\tau) + \lambda\rho_2^2(\tau)][(1-\lambda)\sigma_1^4 + \lambda\sigma_2^4] \\ &+\lambda(1-\lambda)[\rho_1^2(\tau)\sigma_2^2 - \rho_2^2(\tau)\sigma_1^2](\sigma_2^2 - \sigma_1^2). \end{aligned} \tag{2.5}$$

Since $R_2(\tau)$ is clearly different from a linear combination of $R(\tau)^2$ a non-Gaussian mixture process can be differentiated from a Gaussian process.

From the definition of the fourth-order cumulant given in (1.1),

$$C_{42}(\tau) = R_2(\tau) - [(1-\lambda)\sigma_1^2 + \lambda\sigma_2^2]^2 - 2R(\tau)^2. \tag{2.6}$$

As expected, if $\lambda = 1, 0$, (the Gaussian case), then $C_{42}(\tau) = 0$. However, for the non-Gaussian case, when $\sigma_1 = \sigma_2 = 1$,

$$C_{42}(\tau) = 2\lambda(1-\lambda)[\rho_1(\tau) - \rho_2(\tau)]^2. \tag{2.7}$$

The fourth-order spectrum of (2.7) is,

$$C_{42}(\omega) = 2\lambda(1-\lambda)\int_{-\infty}^{\infty}[\rho_1(\tau) - \rho_2(\tau)]^2 e^{-j\omega\tau}d\tau. \tag{2.8}$$

EXAMPLE 1.

Let, $\rho_1(\tau) = e^{-\alpha_1|\tau|}$, and $\rho_2(\tau) = e^{-\alpha_2|\tau|}$ in equation (2.8), then for $\lambda \neq 0, 1$,

$$C'_{42}(\omega) = \frac{C_{42}(\omega)}{2\lambda(1-\lambda)} = \frac{4\alpha_1}{\omega^2 + (2\alpha_1)^2} + \frac{4\alpha_2}{\omega^2 + (2\alpha_2)^2} - \frac{4(\alpha_1+\alpha_2)}{\omega^2 + (\alpha_1+\alpha_2)^2}. \tag{2.9}$$

Equation (2.9) is plotted for positive frequencies in figure 1 for three cases: $(\alpha_1 = 10, \alpha_2 = 1)$,$(\alpha_1 = 10, \alpha_2 = 2)$, and $(\alpha_1 = 10, \alpha_2 = 9)$. The last case is close to zero for all frequencies indicating that the process is almost Gaussian. The other two cases show that the fourth-order spectrum can differentiate between Gaussian and non-Gaussian processes. It is also apparent from the figure that the fourth-order spectrum is negative for some frequencies.

EXAMPLE 2.

Let, $\rho_1(\tau) = e^{-\alpha_1|\tau|}\cos(\omega_0\tau)$, and, $\rho_2(\tau) = e^{-\alpha_2|\tau|}\cos(\omega_0\tau)$, in (2.8), then, for $\lambda \neq 0, 1$,

$$\begin{aligned} C'_{42}(\omega) = \frac{C_{42}(\omega)}{2\lambda(1-\lambda)} &= \frac{2\alpha_1}{\omega^2 + (2\alpha_1)^2} + \frac{\alpha_1}{(\omega - 2\omega_0)^2 + (2\alpha_1)^2} + \frac{\alpha_1}{(\omega + 2\omega_0)^2 + (2\alpha_1)^2} \\ &+ \frac{2\alpha_2}{\omega^2 + (2\alpha_2)^2} + \frac{\alpha_2}{(\omega - 2\omega_0)^2 + (2\alpha_2)^2} + \frac{\alpha_2}{(\omega + 2\omega_0)^2 + (2\alpha_2)^2} \\ &- \frac{2(\alpha_1 + \alpha_2)}{\omega^2 + (\alpha_1 + \alpha_2)^2} - \frac{\alpha_1 + \alpha_2}{(\omega - 2\omega_0)^2 + (\alpha_1 + \alpha_2)^2} - \frac{\alpha_1 + \alpha_2}{(\omega + 2\omega_0)^2 + (\alpha_1 + \alpha_2)^2}. \end{aligned} \tag{2.10}$$

Equation (2.10) is plotted in figure 2 for the same conditions as given in example 1.

The fourth-order moment, for the other special case mentioned in the introduction, is

$$C_{43}(\tau) = 3\lambda(1-\lambda)[\rho_2(\tau)\sigma_1^2(\sigma_1^2 - \sigma_2^2) + \rho_1(\tau)\sigma_2^2(\sigma_2^2 - \sigma_1^2)].$$

If $\sigma_1 = \sigma_2$, then $C_{43}(\tau) = 0$. This case will not be considered further.

3. Sinusoidal Mixture Processes

A problem that occurs in underwater acoustics can be stated as follows: given s(t), -T≤t≤T, determine if s(t) represents a sum of two sinusoids or a mixture of two sinusoids. A more complicated problem arises from creeping wave backscatter. This case is usually discussed in the context of target classification from resonances[18]. Here, however, the backscattered waveform is idealized as a mixture of two sinusoids in order to show the classification potential of fourth-order spectra. Whereas, both a sum and a mixture of two sinusoids will have fourth-order spectra, only the mixture represents the idealized backscattered waveform. Therefore, the fourth-order spectrum gives additional information about the physical phenomenon that can be utilized in target classification.

Nevertheless, it is interesting that the idealized problem can be solved by fourth- order spectra. On the other hand, second-order spectra cannot differentiate between a sum and a mixture, since both could have the same spectra. However, there may be amplitude differences, but this information is usually not enough to make a determination.

It will be assumed that the data are ergodic so that the time autocorrelation function $\Re(\tau)$ and the time fourth-order function $\Re_2(\tau)$ of the random process equals the corresponding statistical autocorrelation function and statistical fourth-order function, respectively[19].

a) Sum of Sinusoids

Let,

$$s(t) = \frac{a_1}{\sqrt{2}}\sin(\omega_1 t + \phi_1) + \frac{a_2}{\sqrt{2}}\sin(\omega_2 t + \phi_2), \tag{3.1}$$

where, $\omega_1 \neq \omega_2$, $-T \leq t \leq T$, and the parameters ϕ_1 and ϕ_2 are random variables uniformly distributed between 0 and 2π.

The time autocorrelation function,

$$\Re(\tau) = \lim_{T\to\infty} \frac{1}{2(T-|\tau|)} \int_{-T+|\tau|}^{T-|\tau|} s(t)s(t+\tau)dt,$$

reduces to,

$$\Re(\tau) = \frac{a_1^2}{4}\cos(\omega_1\tau) + \frac{a_2^2}{4}\cos(\omega_2\tau) = R(\tau). \tag{3.2}$$

Its corresponding spectrum is given in (3.3),

$$S(\omega) = \frac{2\pi a_1^2}{4}\frac{\delta(\omega-\omega_1)+\delta(\omega+\omega_1)}{2} + \frac{2\pi a_2^2}{4}\frac{\delta(\omega-\omega_2)+\delta(\omega+\omega_2)}{2}. \tag{3.3}$$

Similarly, the time fourth-order function,

$$\Re_2(\tau) = \lim_{T\to\infty} \frac{1}{2(T-|\tau|)} \int_{-T+|\tau|}^{T-|\tau|} s(t)^2 s(t+\tau)^2 dt,$$

for (3.1), reduces to,

$$\Re_2(\tau) = \frac{a_1^4 + 2a_1^2a_2^2 + a_2^2}{16} + \frac{a_1^4}{32}\cos(2\omega_1\tau) + \frac{a_2^4}{32}\cos(2\omega_2\tau) + \frac{a_1^2a_2^2}{4}\cos(\omega_1\tau)\cos(\omega_2\tau) = R_2(\tau). \tag{3.4}$$

Notice that the spectrum of $R_2(\tau)$ will have 5 positive frequencies. However, the fourth-order cumulant is desired. It reduces to,

$$C_{42}(\tau) = -\frac{a_1^4 + a_2^4}{16} - \frac{a_1^4}{32}\cos(2\omega_1\tau) - \frac{a_2^4}{32}\cos(2\omega_2\tau),$$

and its corresponding spectrum follows as,

$$C_{42}(\omega) = -\frac{2\pi(a_1^4 + a_2^4)}{16}\delta(\omega) - \frac{2\pi a_1^4}{32}\frac{\delta(\omega-2\omega_1)+\delta(\omega+2\omega_1)}{2} - \frac{2\pi a_2^4}{32}\frac{\delta(\omega-2\omega_2)+\delta(\omega+2\omega_2)}{2}. \tag{3.5}$$

Therefore, the fourth-order cumulant spectrum for (3.1) contains only 3 positive frequencies located at $0, 2\omega_1$, and $2\omega_2$. The frequency locations for the fourth-order cumulant of the mixture process will be different. This represents a means to differentiate a sum from a mixture.

b) Mixture of Sinusoids

Let,

$$s(t) = a_1 \sin(\omega_1 t + \phi_1) h_1(t) + a_2 \sin(\omega_2 t + \phi_2) h_2(t), \tag{3.6}$$

represent the sinusoidal mixture process over the interval, $-T_L \leq t \leq T_U$, where $T_L \geq 0$, and $T_U \geq 0$. The functions $h_1(t)$ and $h_2(t)$ are defined in the following way,

$$h_1(t) = u(t + T_L) - u(t - T_1),$$

and

$$h_2(t) = u(t - T_1) - u(t - T_U),$$

where u() is the unit step function. These functions also have the property that $h_i^2() = h_i()$, i=1,2, which will be used later.

The time autocorrelation function of (3.6),

$$\Re(\tau) = \lim_{T_L, T_U \to \infty} \frac{1}{T_L + T_U - 2|\tau|} \int_{-T_L + |\tau|}^{T_U - |\tau|} s(t) s(t + \tau) dt,$$

reduces to, for $\tau > 0$,

$$\Re(\tau) = \lim_{T_L, T_U \to \infty} \left[\frac{a_1^2}{2} \cos(\omega_1 \tau) \left(\frac{T_L + T_1}{T_L + T_U} \right) \left(\frac{1 - \frac{|\tau|}{T_L + T_1}}{1 - \frac{2|\tau|}{T_L + T_U}} \right) \right.$$

$$\left. + \frac{a_2^2}{2} \cos(\omega_2 \tau) \left(1 - \frac{T_L + T_1}{T_L + T_U} \right) \left(\frac{1 - \frac{2|\tau|}{T_U - T_1}}{1 - \frac{2|\tau|}{T_L + T_U}} \right) \right].$$

A similar expression is obtained for $\tau < 0$. These expressions converge to the following,

$$\Re(\tau) = (1 - \lambda) \frac{a_2^2}{2} \cos(\omega_2 \tau) + \lambda \frac{a_1^2}{2} \cos(\omega_1 \tau), \tag{3.7}$$

where, $\frac{T_L + T_1}{T_L + T_U} \to \lambda$. If $\lambda = \frac{1}{2}$, then (3.7) and (3.2) will have identical spectra.

The fourth-order moment can be shown to converge to the following result,

$$\Re_2(\tau) = \frac{a_1^4 \lambda + a_2^4 (1 - \lambda)}{4} + \frac{a_2^4 (1 - \lambda)}{8} \cos(2\omega_2 \tau) + \frac{a_1^4 \lambda}{8} \cos(2\omega_1 \tau). \tag{3.8}$$

Notice that only 3 positive frequencies are present in (3.8). The fourth-order cumulant is given by,

$$C_{42}(\tau) = \frac{1}{4}\lambda(1-\lambda)(a_1^2 - a_2^2)^2 - \frac{a_1^4\lambda^2}{4} - \frac{a_2^4(1-\lambda)^2}{4}$$
$$+\frac{a_1^4\lambda(1-2\lambda)}{8}\cos(2\omega_1\tau) - \frac{a_2^4(1-\lambda)(1-2\lambda)}{8}\cos(2\omega_2\tau) - a_1^2a_2^2\lambda(1-\lambda)\cos(\omega_1\tau)\cos(\omega_2\tau).$$

If, $a_1 = a_2 = 1$, and $\lambda = \frac{1}{2}$, the fourth-order cumulant reduces to,

$$C_{42}(\tau) = -\frac{1}{8} - \frac{1}{4}\cos(\omega_1\tau)\cos(\omega_2\tau), \tag{3.9}$$

and its corresponding spectrum,

$$C_{42}(\omega) = -\frac{2\pi}{8}\delta(\omega) - \frac{2\pi}{8}\frac{\delta[\omega-(\omega_2-\omega_1)] + \delta[\omega+(\omega_2-\omega_1)]}{2} - \frac{2\pi}{8}\frac{\delta[\omega-(\omega_2+\omega_1)] + \delta[\omega+(\omega_2+\omega_1)]}{2}, \tag{3.10}$$

is clearly different from the spectrum in (3.5), for the same conditions.

EXAMPLE 3.

(Sum of Sinusoids)

Figure 3 represents the spectrum (in db) of (3.2) for $a_1 = a_2 = 1$, and $\lambda = \frac{1}{2}$. The corresponding fourth-order cumulant spectrum (for a linear scale) is shown in figure 4.

(Mixture of Sinusoids)

The spectrum of (3.7) is shown in figure 5, for $a_1 = a_2 = 1$, and $\lambda = \frac{1}{2}$. In this case the spectra of figures 3 and 5 are identical. However, the fourth-order cumulant spectrum (3.10) as shown in figure 6 is clearly different from figure 4. Therefore, the sinusoidal mixture process can be differentiated from a sum of sinusoids, based on the fourth-order cumulant spectrum.

4. Modulated Processes

Modulated processes arise in communications as well as sonar and radar applications. Here the following general modulated problem is treated. Let,

$$x(t) = a(t)s(t), \tag{4.1}$$

be the modulated process. Where a(t) and s(t) are both sero mean and mutually independent processes. Equation (4.1) is in general a non-Gaussian process even if both a(t) and s(t) are Gaussian processes. Its autocorrelation function,

$$R_x(\tau) = E[x(t)x(t+\tau)] = E[a(t)a(t+\tau)]E[s(t)s(t+\tau)] = R_a(\tau)R_s(\tau), \tag{4.2}$$

is a product of two autocorrelation functions. Therefore, its spectrum will be a convolution of the two spectra. If $z(t)$ is the information bearing component then the modulating component $a(t)$ tends to interfere with the reception of $z(t)$.

Let $a(t)$ be a Gaussian process. Then the fourth-order cumulant of equation (4.1) reduces to

$$C_{42}(\tau) = \mathrm{Var}^2[a(t)]\left[E[z(t)^2 z(t+\tau)^2] - \mathrm{Var}^2[z(t)]\right] + 2R_a(\tau)^2\left[E[z(t)^2 z(t+\tau)^2] - R_z(\tau)^2\right]. \tag{4.3}$$

Notice that (4.3) is not a simple product of autocorrelation functions. Therefore, its spectrum will be different from the spectrum of (4.2).

Now let $z(t)$ be Gaussian also. Then (4.3) reduces to,

$$C_{42}(\tau) = 2\mathrm{Var}^2[a(t)]R_z(\tau)^2 + 2\mathrm{Var}^2[z(t)]R_a(\tau)^2 + R_a(\tau)^2 R_z(\tau)^2, \tag{4.4}$$

which can be seen to separate the two spectra into a sum and a product rather than just a product.

EXAMPLE 4.

Let,

$$x(t) = a(t)\cos(\omega_0 t + \phi),$$

where, ϕ is a random parameter uniformly distributed between 0 and 2π. Then,

$$R_x(\tau) = R_a(\tau)\frac{\cos(\omega_0\tau)}{2}.$$

If,

$$R_a(\tau) = e^{-\alpha|\tau|},$$

then the spectrum of $R_x(\tau)$ is,

$$S_x(\omega) = \frac{\alpha}{(\omega-\omega_0)^2+\alpha^2} + \frac{\alpha}{(\omega+\omega_0)^2+\alpha^2}. \tag{4.5}$$

If $\alpha \to \infty$, then $S_x(\omega) \to 0$. This means that the sinusoidal frequency ω_0 will not be observed in the spectrum of $R_x(\tau)$.

However, the fourth-order cumulant,

$$C_{42}(\tau) = \frac{1}{4}R_a(\tau)^2 + \frac{\mathrm{Var}^2[a(t)]}{8}\cos(2\omega_0\tau),$$

has the following spectrum,

$$C_{42}(\omega) = \frac{\alpha}{\omega^2+(2\alpha)^2} + \frac{2\pi\mathrm{Var}^2[a(t)]}{8}\,\frac{\delta(\omega-2\omega_0)+\delta(\omega+2\omega_0)}{2}. \tag{4.6}$$

Now, the sinusoidal frequency $2\omega_0$ is seen in (4.6) even as $\alpha \to \infty$.

(Simulation: Amplitude Modulated Sinusoid)

The spectrum (in db) of a sinusoid, amplitude modulated by white noise, is shown in figure 7. The sinusoidal frequency is not discernable in this figure because of the interference of the modulation as predicted in (4.5). However, the fourth-order spectrum in figure 8 (plotted on a linear scale) reveals the underlying sinusoid but at twice its frequency. This result is predicted in (4.6).

EXAMPLE 5. (SONAR/RADAR APPLICATION)

Let,

$$y(t) = \sum_{n=-k}^{k} h(t - nT_p)e^{j(\omega_n t + \theta_n)},$$

$-\infty \leq t \leq \infty$, be the pulse coded transmitted signal of length, $2kT_p + T$. The pulses are defined as follows,

$$h(t - nT_p) = u(t - nT_p + \frac{T}{2}) - u(t - nT_p - \frac{T}{2}),$$

where u() is the unit step function. The parameters, T and T_p, for $T \leq T_p$, represent the pulse width and pulse repetition interval, respectively.

The received signal is assumed to be of the form,

$$x'(t) = a(t - \frac{T_R}{2}) \sum_{n=-k}^{k} h(t - nT_p - T_R)e^{j[(\omega_n + \omega_{nd})(t - T_R) + \theta_n + \phi]},$$

where, T_R is the range of the target, ω_{nd} is the Doppler shift radian frequency associated with each transmitted radian frequency ω_n, θ_n is the phase of the nth transmitted pulse, and ϕ is a random phase angle uniformly distributed between 0 and 2π. The real stochastic Gaussian modulating function a(t) represents a time fluctuating target[20]. This model usually applies for transmitted signals of long length.

Let,

$$y'(t) = \sum_{n=-k}^{k} h(t - nT_p - T_R')e^{j[\omega_n(t - T_R') + \theta_n]},$$

be the adjusted transmitted signal. Where T_R' is a parameter that is adjusted in order to search for the true range. For simplicity, it will be assumed that $T_R' = T_R$, i.e., the range is known.

The envelope of the received signal is defined as follows,

$$x(t) = x'(t)y'^*(t),$$

where the asterisk is the complex conjugate.

Therefore, since the pulses are disjoint, i.e.,

$$h(t-n_1T_p)h(t-n_2T_p) = \begin{cases} h(t\text{-}n_1T_p), & if\ n_1 = n_2, \\ 0, & if\ n_1 \neq n_2, \end{cases}$$

the envelope of the received signal is,

$$x(t) = a(t - \frac{T_R}{2}) \sum_{n=-k}^{k} h(t - nT_p - T_R)e^{j[\omega_{nd}(t-T_R)+\phi]}. \tag{4.7}$$

The objective of this example is to compare the autocorrelation function of (4.7) and its corresponding spectrum with the fourth-order cumulant function of (4.7) and its corresponding spectrum.

The autocorrelation function is given by the following expression,

$$E[x(t_1)x^*(t_2)] = R_a(t_2 - t_1) \sum_{n_1=-k}^{k} \sum_{n_2=-k}^{k} h(t_1 - n_1T_p - T_R)h(t_2 - n_2T_p - T_R)e^{j(\omega_{n_1d}t_1 - \omega_{n_2d}t_2)}, \tag{4.8}$$

where,

$$R_a(t_2 - t_1) = E[a(t_1 - \frac{T_R}{2})a(t_2 - \frac{T_R}{2})],$$

is a stationary autocorrelation function.

Therefore, since (4.7) is not in general a stationary process, the spectrum of (4.8) is defined as a two-dimensional Fourier transform of the autocorrelation function,

$$S(T_R, \omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t_1)x^*(t_2)]e^{-j(\omega_1t_1+\omega_2t_2)}dt_1dt_2. \tag{4.9}$$

To simplify the evaluation of (4.9) the modulating function will be assumed to be white noise, i.e., $R_a(t_2 - t_1) = \delta(t_2 - t_1)$. Therefore, the spectrum of (4.8) reduces to

$$S(T_R, \omega_1, \omega_2) = Te^{-j(\omega_1+\omega_2)T_R}\left[\frac{\sin[(\omega_1+\omega_2)\frac{T}{2}]}{[(\omega_1+\omega_2)\frac{T}{2}]}\frac{\sin[(\omega_1+\omega_2)(k+\frac{1}{2})T_p]}{\sin[(\omega_1+\omega_2)\frac{T_p}{2}]}\right]. \tag{4.10}$$

The spectrum (4.10) is concentrated along the line, $\omega_1 = -\omega_2$ with the same value, T(2k+1), for all frequencies. Therefore, if $\omega_1 = -\omega_2$, then, $S(T_R, \omega) = T(2k+1)$, $\forall\omega$. This clearly shows the interfering effect of the modulating white noise.

The fourth-order cumulant function reduces to the following result,

$$C_{42}(t_1, t_2) = \mathrm{Var}^2[a] \sum_{n_1=-k}^{k}\sum_{n_2=-k}^{k} h(t_1 - n_1T_p - T_R)h(t_2 - n_2T_p - T_R)e^{j2(\omega_{n_1d}t_1 - \omega_{n_2d}t_2)}, \tag{4.11}$$

where the Gaussian assumption for the modulating function was used to obtain (4.11).

Similarly, the fourth-order cumulant spectrum is defined as a two-dimensional Fourier transform,

$$C_{42}(T_R, \omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{42}(t_1, t_2) e^{-j(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2, \tag{4.12}$$

which reduces to,

$$C_{42}(T_R, \omega_1, \omega_2) = \mathrm{Var}^2[a] T^2 \sum_{n_1=-k}^{k} \sum_{n_2=-k}^{k} e^{-j[\omega_1 + \omega_2 - 2(\omega_{n_1 d} - \omega_{n_2 d})][(n_1+n_2)T_p + 2T_R]}$$

$$\left[\frac{\sin[(\omega_1 - 2\omega_{n_1 d})\frac{T}{2}]}{(\omega_1 - 2\omega_{n_1 d})\frac{T}{2}} \frac{\sin[(\omega_2 + 2\omega_{n_2 d})\frac{T}{2}]}{(\omega_2 + \omega_{n_2 d})\frac{T}{2}} \right]. \tag{4.13}$$

Again, the spectrum is concentrated along the line, $\omega_1 = -\omega_2$, but now with peaks at twice the Doppler shift frequencies. To see this more clearly, let $\omega_1 = -\omega_2 = \omega$, and $\omega_{nd} = \omega_d, \forall n$, i.e., all the Doppler shift frequencies are the same, then

$$C_{42}(T_R, \omega) = \mathrm{Var}^2[a] T^2 (2k+1)^2 \left[\frac{\sin[(\omega - 2\omega_d)\frac{T}{2}]}{(\omega - 2\omega_d)\frac{T}{2}} \right]^2. \tag{4.14}$$

It is clear that (4.13) and (4.14) are not interfered with by the modulating function, except for the constant representing its variance. Therefore, the fourth-order cumulant spectrum of the received signal will be discernable. Whereas, the spectrum of (4.8) may not be discernable depending on the modulating function a(t) as shown in example 4.

5. Conclusions

It has been shown that for non-Gaussian processes, modeled as mixture and modulated processes, higher-order spectra can extract information that second-order spectra cannot. Namely, the fourth-order cumulant spectrum can differentiate between purely Gaussian processes and mixture non-Gaussian processes. This was demonstrated by several examples. Moreover, the fourth-order cumulant spectra can differentiate between a sum of sinusoids and a mixture of sinusoids. This was demonstrated by simulation.

An important practical application of fourth-order spectra showed that target information, from an active sonar return, could be extracted . Specifically, the fourth-order spectrum extracted Doppler shift frequencies from an amplitude modulated coded pulse train return, whereas, the spectrum was unable to do so.

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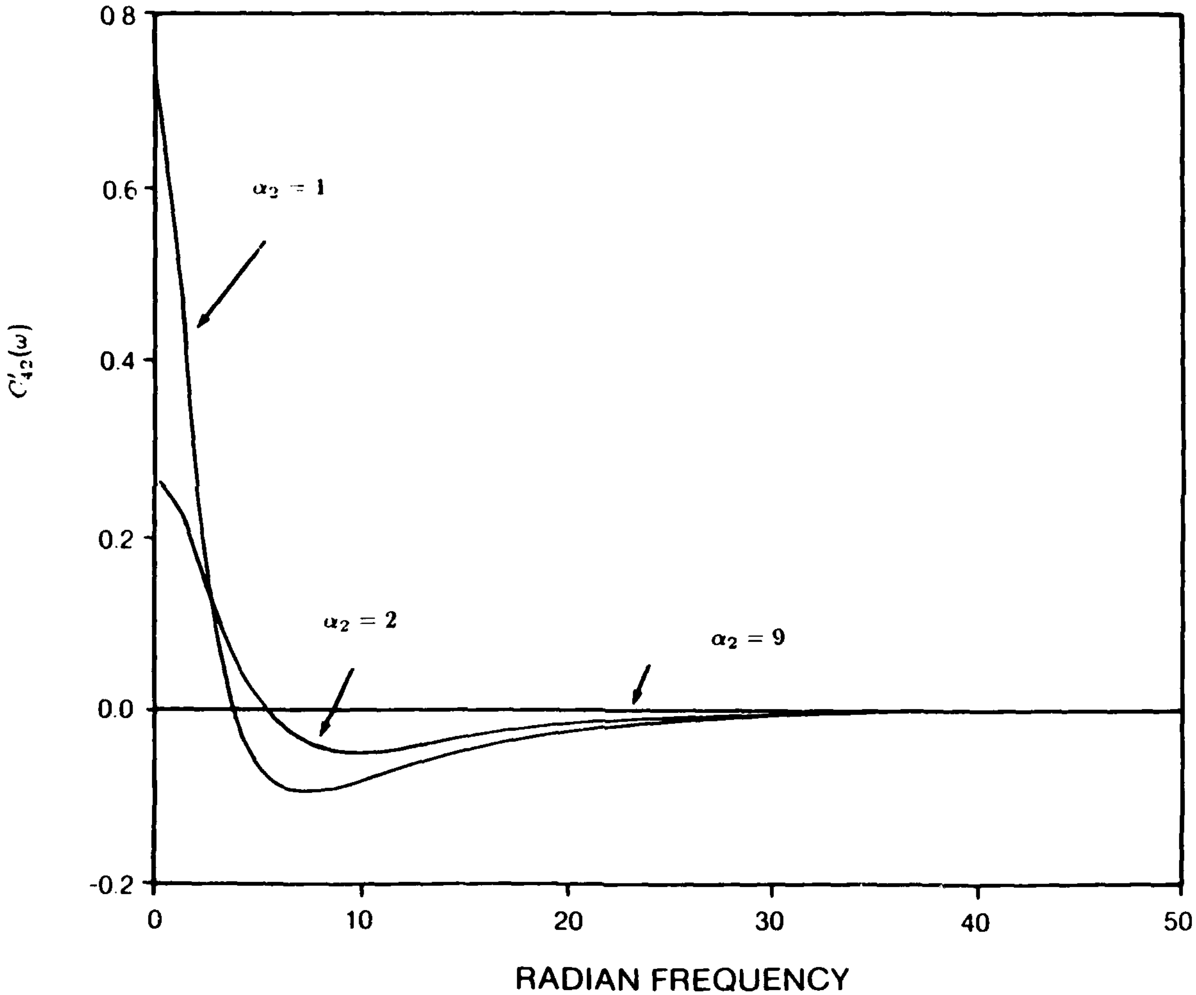

FIGURE 1. Fourth-Order Spectrum: α_1 10.

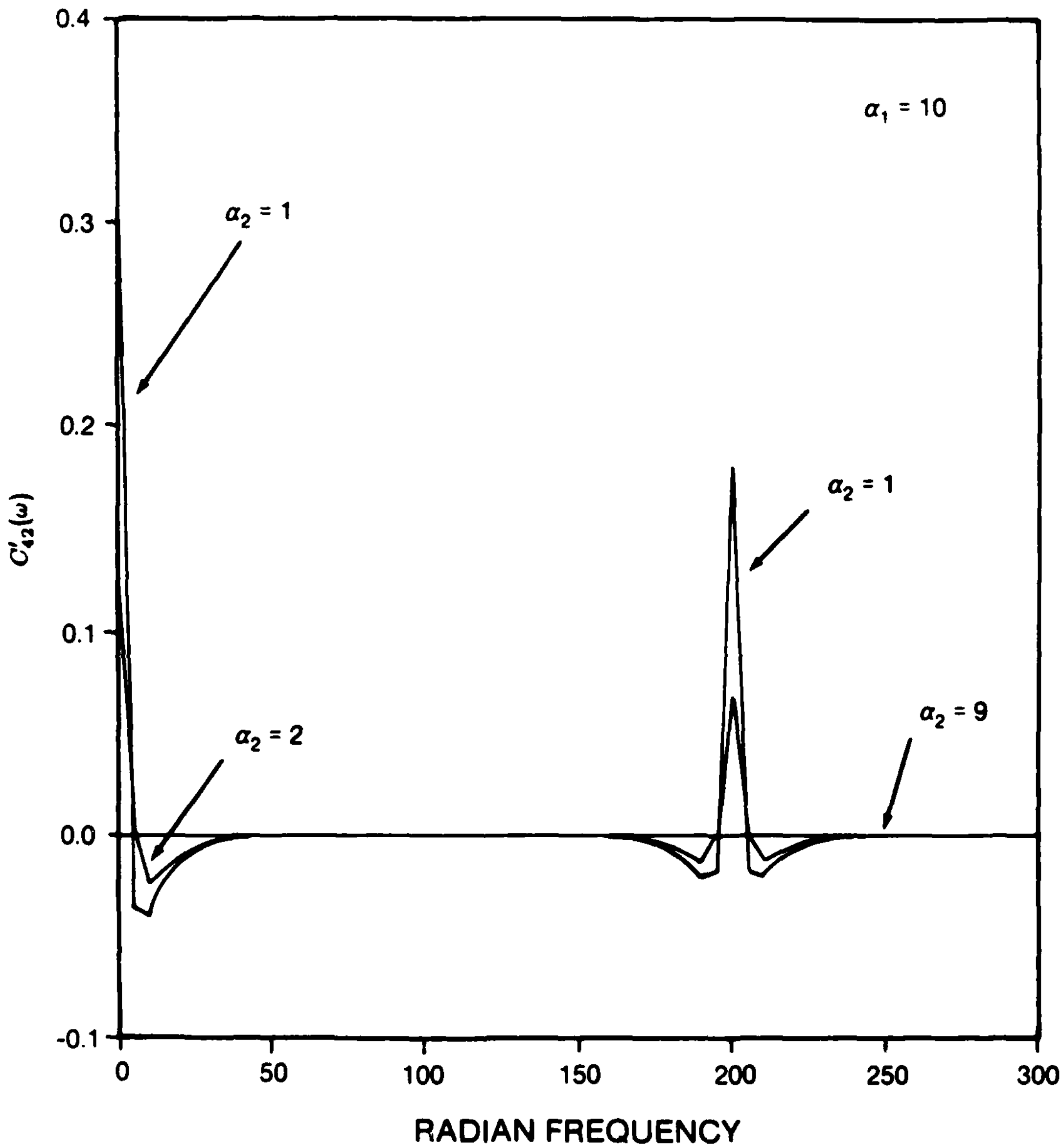

FIGURE 2. Example 2, Fourth-Order Spectrum: $\alpha_1 = 10$.

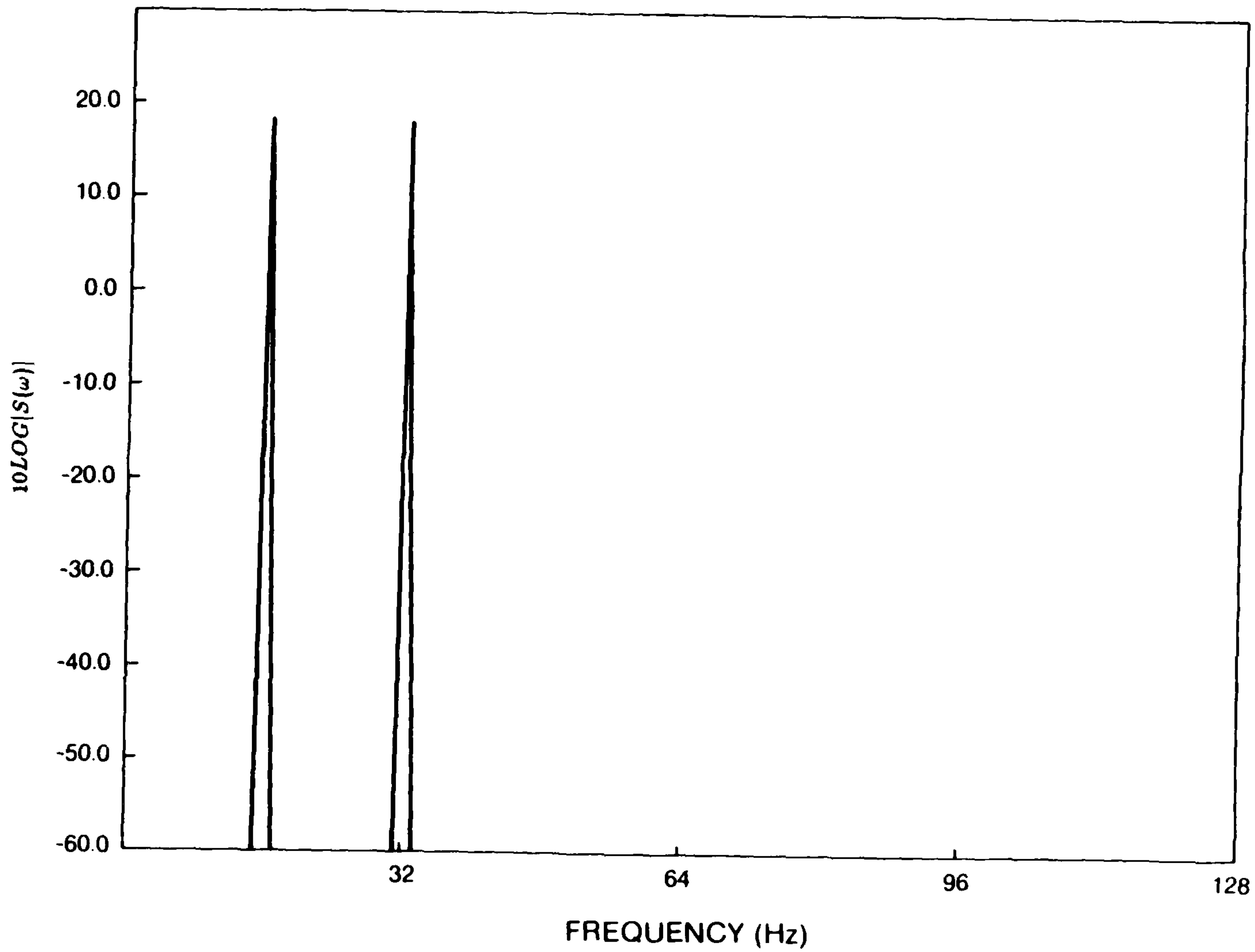

FIGURE 3. Sum of Sinusoids Spectrum.

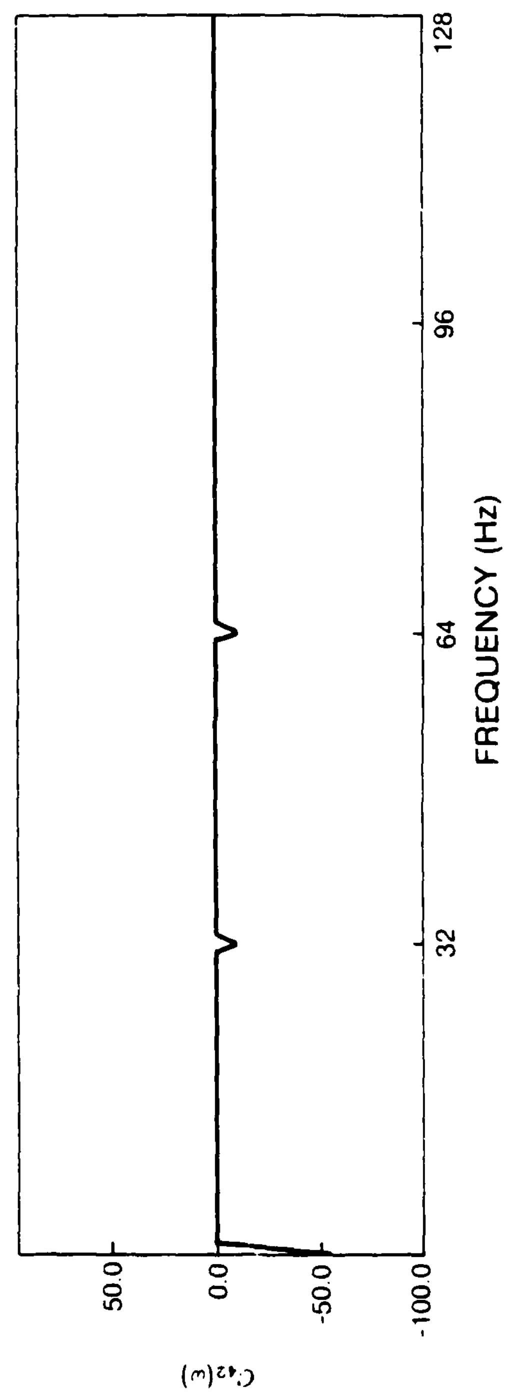

FIGURE 4. Sum of Sinusoids Fourth-Order Spectrum.

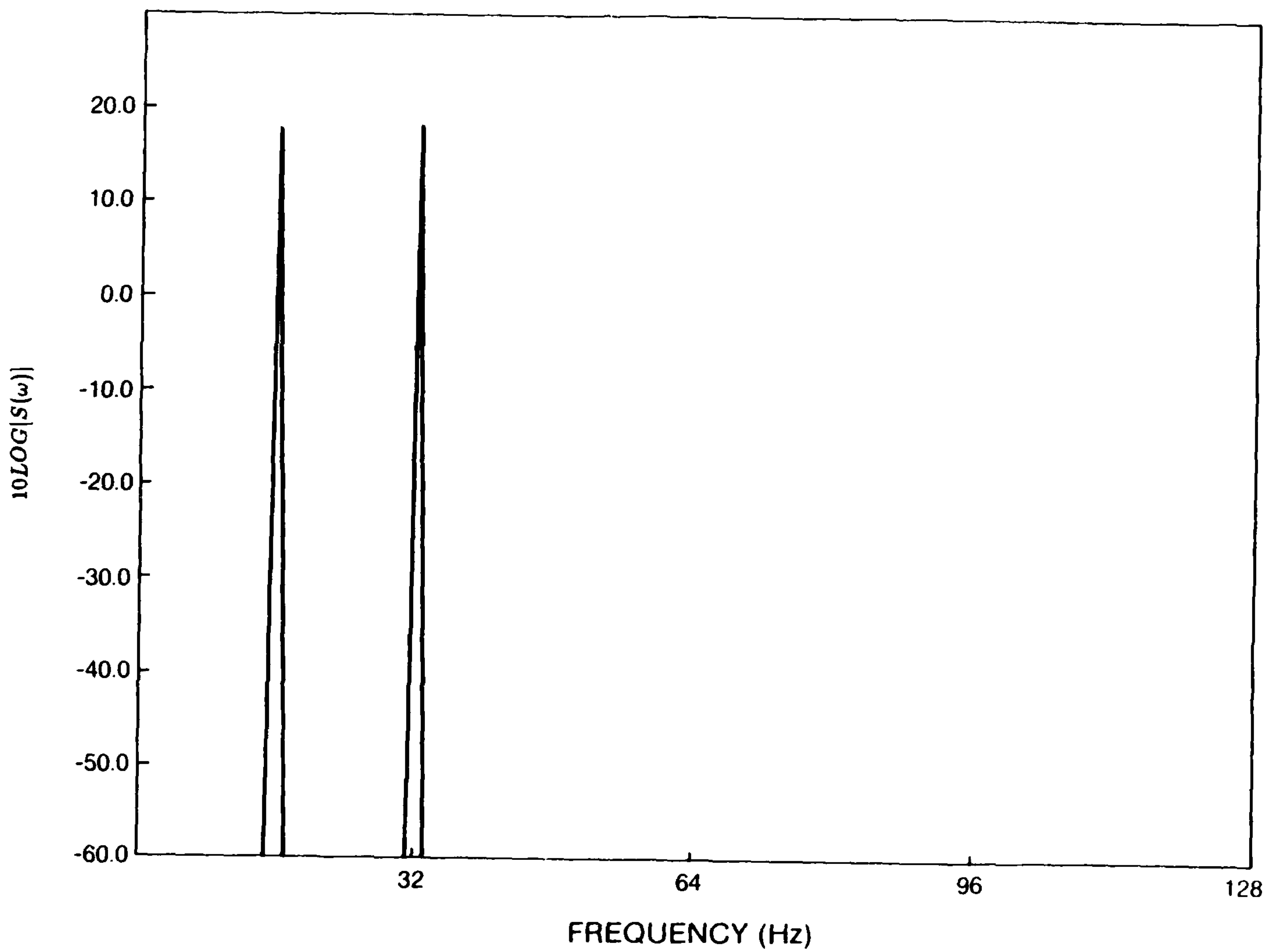

FIGURE 5. Mixture of Sinusoids Spectrum.

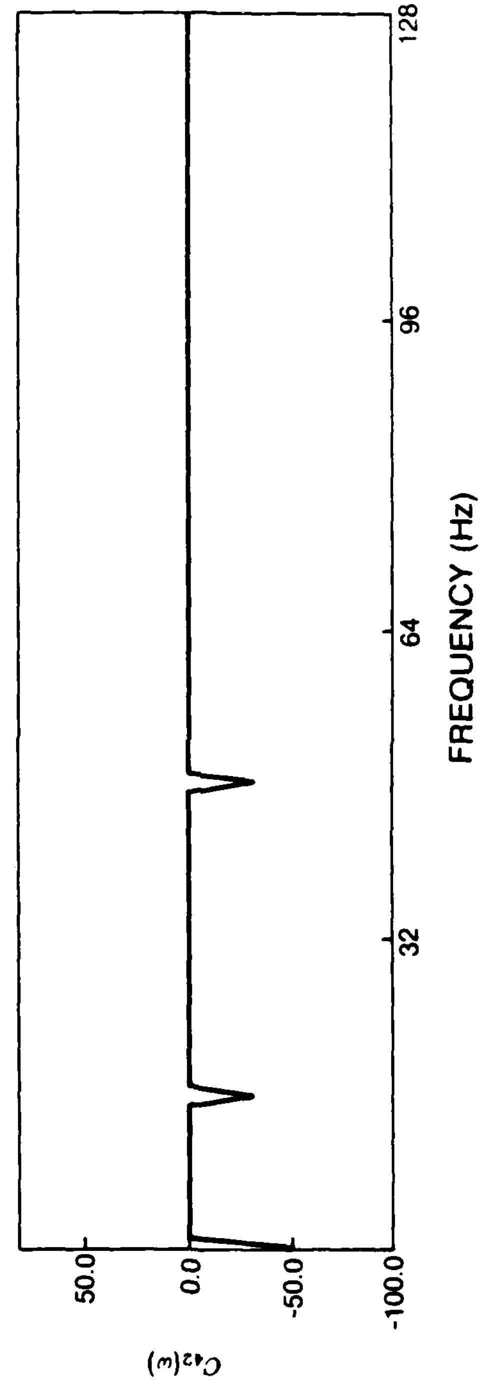

FIGURE 6. Mixture of Sinusoids Fourth-Order Spectrum.

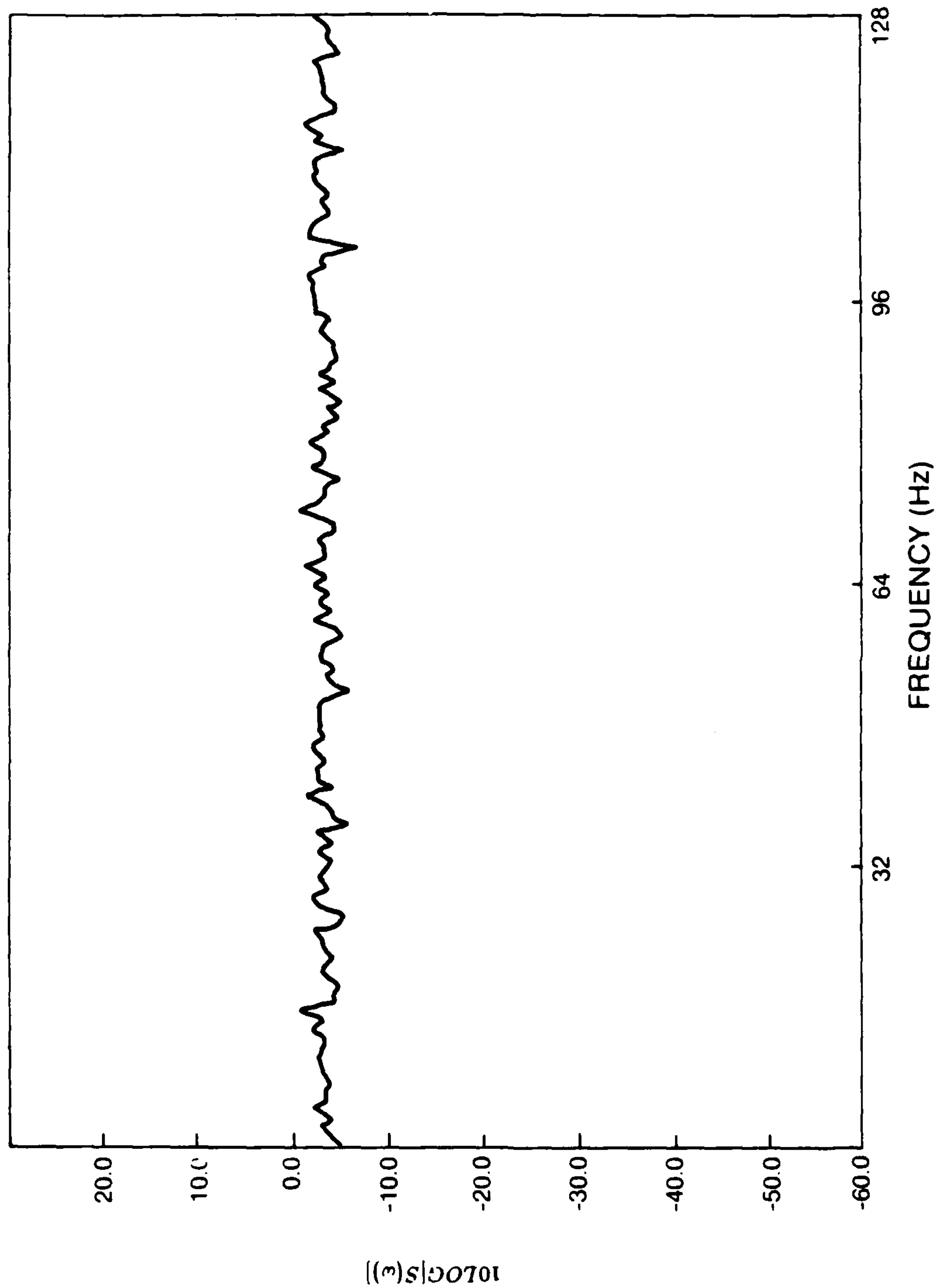

FIGURE 7. Modulated Spectrum.

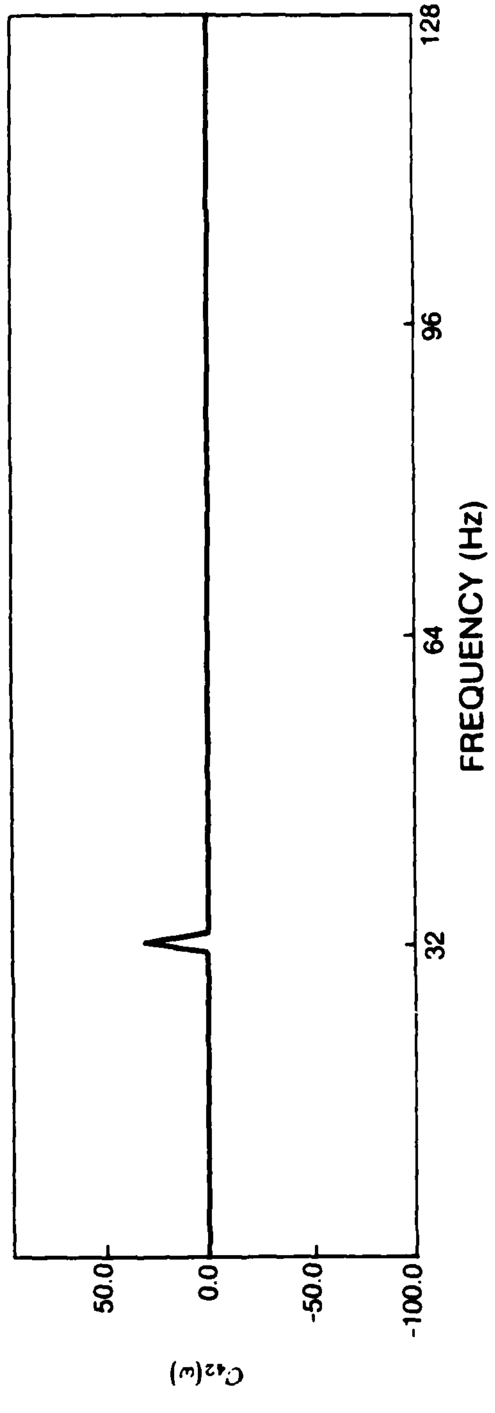

FIGURE 8. Modulated Fourth-Order Spectrum.

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